

Phase transitions and perfectness of fluids in weakly coupled real scalar field theories

Jiunn-Wei Chen^{1,2}, Mei Huang³, Yen-Han Li¹, Eiji Nakano¹, Di-Lun Yang¹

¹ Department of Physics and Center for Theoretical Sciences, National Taiwan University, Taipei 10617

² CTP, Massachusetts Institute for Technology, Cambridge, MA 02139 and

³ Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049

We calculate the ratio η/s , the shear viscosity (η) to entropy density (s), which characterizes how perfect a fluid is, in weakly coupled real scalar field theories with different types of phase transitions. The mean-field results of the η/s behaviors agree with the empirical observations in atomic and molecular systems such as H₂O, He, N, and all the matters with data available in the NIST database. These behaviors are expected to be the same in N component scalar theories with an $O(N)$ symmetry. We speculate these η/s behaviors are general properties of fluid shared by QCD and cold atoms. Finally, we clarify some issues regarding counterexamples of the conjectured universal bound $\eta/s \geq 1/4\pi$ found in Refs. [16, 17].

PACS numbers: 11.10.Wx, 51.20.+d, 12.38.Mh

Quantum chromodynamics (QCD) is believed to undergo rapid transitions from hadronic phases to a quark-gluon plasma (QGP) phase at high temperature T and to quark matter phases at high quark chemical potential μ (see [1, 2, 3] for reviews). Lattice results show that the phase transition of the hadronic matter to QGP at finite T with $\mu = 0$ is likely a crossover [4]. At finite μ and $T = 0$, there is no reliable lattice result due to the severe fermion sign problem. However, arguments based on a variety of models show that the phase transition is of first-order. This first-order phase transition turns into a crossover at smaller μ and finite T at the QCD critical end-point (CEP) [5]. There are lots of interests in the CEP. Recently, it was proposed to probe the CEP by using the ratio of shear viscosity η to the entropy density s of QCD [6].

Shear viscosity η characterizes how strongly particles interact and move collectively in a many-body system. In general, the stronger the interparticle interaction, the smaller the shear viscosity (here η is normalized by the density). It was conjectured [7] that no matter how strong the particle interaction is, η/s has a universal minimum bound $1/4\pi$ in any system. This bound is motivated by the uncertainty principle and is found to be saturated for a large class of strongly interacting quantum field theories whose dual descriptions in string theory involve black holes in anti-de Sitter space [7].

There are two important questions regarding this conjecture. The first one is: Is this η/s bound truly universal? By definition, there is no proof of this conjecture yet. From experimental observations, the bound is well satisfied in matters like H₂O, N and superfluid He [7]. For cold fermionic atoms with an infinite scattering length (the unitarity limit), the bound is satisfied but η/s is close to the bound near the phase transition temperature T_c [8]. Similar behavior is found for QCD at $\mu = 0$ [9, 10]. Relativistic heavy ion collisions (RHIC) [11, 12, 13, 14] and lattice computations of a gluon plasma [15]) suggest that η/s of QCD is close to the minimum bound at just above T_c . But one cannot conclude whether the bound is violated based on current

precision. On the other hand, interesting counterexamples of the bound have been constructed [16, 17] and a possible modification of the bound due to some string excitations in the dual theory [18, 19, 20].

The second question is: What is the general behavior of η/s , especially for QCD and cold atom systems which are of high interests but not well known? The answer to this question is interesting on its own. It also has interesting applications. For example, locating QCD CEP by η/s requires this information. Also, knowing when a system will reach its minimum η/s is important for testing the minimum bound conjecture. By now it is known that several different systems have qualitatively the same η/s behaviors. For example, H₂O, N, and He all have the minimum η/s near the liquid-gas phase transition temperature. The same is true with QCD at $\mu = 0$ [9, 10] and near the nuclear liquid-gas phase transition [21]. It is also true in cold atoms in the unitarity limit [8]. It would certainly be interesting to further map out the detailed η/s structure in the phase diagram of QCD and other systems.

In this letter, we address the above two questions at the same time. We study how η/s behaves in the simplest field theory—a real scalar field theory. The resulting η/s in first-, second-order phase transitions and crossover behaves the same way as in H₂O, N, He, and all the matters with data available in the NIST database [9, 21, 22]. We argue that this agreement might hold when the theory is generalized to N components with an $O(N)$ symmetry, which is the low energy effective field theory of a big class of systems. We then speculate these behaviors might be general properties of fluid which are shared by QCD and cold atoms. Furthermore, this simple theory allows us to clarify some issues regarding the counterexamples of the η/s bound [16, 17] in a more transparent way.

We will study a real scalar theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}a\phi^2 - \frac{1}{4}b\phi^4 - \frac{1}{6}c\phi^6. \quad (1)$$

This theory is invariant under $\phi \rightarrow -\phi$ and has a Z_2 symmetry. There could be two additional terms with

dimension six: $\phi^3\partial^2\phi$, $\phi\partial^2\partial^2\phi$ (the other terms are related to these ones by integration by parts). These terms can be removed by field redefinition or, equivalently, by applying the equation of motion. The inclusion of the dimension six terms shows that this is an effective field theory, which is valid under the cut-off scale $1/\sqrt{c}$ and is renormalized order by order in the momentum expansion $p\sqrt{c}$, p being a typical momentum scale in the problem. a , b , and c are renormalized quantities and the counter-term Lagrangian is not shown. The renormalization condition is that the counterterms do not change the particle mass and the four- and six-point couplings at threshold. We will discuss the following cases: 1) $c = 0$, $b > 0$, $a > 0$, the system is always in the symmetric phase. 2) $c = 0$, $b > 0$, $a < 0$, the vacuum at $T = 0$ breaks the Z_2 symmetry spontaneously. However, the symmetry is restored at higher T with a second-order phase transition. 3) Adding an explicit symmetry breaking term $\delta L = H\phi$ to the Lagrangian of 2) to model a crossover. 4) $c > 0$, $b < 0$, $a > 0$, the broken symmetry is restored at high T with a first-order phase transition.

We will focus on the case of weak coupling and compute the mean-field effective potential via the standard Cornwall–Jackiw–Tomboulis (CJT) formalism [23] which has the one-particle irreducible diagrams included self-consistently. The effective potential in the CJT formalism reads [24]

$$V[\bar{\phi}, S] = \frac{1}{2} \int_K [\ln S^{-1}(K) + S_0^{-1}(K) S(K) - 1] + V_2[\bar{\phi}, S] + U(\bar{\phi}), \quad (2)$$

where $U(\bar{\phi}) = a/2 \bar{\phi}^2 + b/4 \bar{\phi}^4 + c/6 \bar{\phi}^6$ is the tree-level potential, and $S(S_0)$ is the full(tree-level) propagator:

$$S^{-1}(K, \bar{\phi}) = -K^2 + m^2(\bar{\phi}), \quad S_0^{-1}(K, \bar{\phi}) = -K^2 + m_0^2(\bar{\phi}), \quad (3)$$

with the tree-level mass $m_0^2 = a + 3b \bar{\phi}^2 + 5c \bar{\phi}^4$.

In this work, we neglect the non-tadpole type loop diagrams. This “Hartree approximation” is good when $T \gg |b|^{1/2} T_c$ for $c = 0$ and when $T \gg |a|^{1/2}$ for $c > 0$. Thus, to the order we are working, the 2 PI potential V_2 only includes

$$V_2[\bar{\phi}, S] = \left(\frac{3b}{4} + \frac{15c}{2} \bar{\phi}^2 \right) L(\bar{\phi})^2 + \frac{15}{6} c L(\bar{\phi})^3, \quad (4)$$

where $L(\bar{\phi}) = \int_K S(K, \bar{\phi})$. The self-consistent one- and two-point Green’s functions satisfy

$$\frac{\delta V}{\delta \bar{\phi}} \bigg|_{\bar{\phi}=\phi_0, S=S(\phi_0)} \equiv 0, \quad \frac{\delta V}{\delta S} \bigg|_{\bar{\phi}=\phi_0, S=S(\phi_0)} \equiv 0. \quad (5)$$

This allows us to solve ϕ_0 and m through the coupled equations:

$$\begin{aligned} \phi_0 (a + b\phi_0^2 + c\phi_0^4 + (3b + 10c\phi_0^2)L(\phi_0) + 15cL(\phi_0)^2) &= 0, \\ m^2 - m_0^2 &= 3(b + 10c\phi_0^2)L(\phi_0) + 15cL(\phi_0)^2. \end{aligned} \quad (6)$$

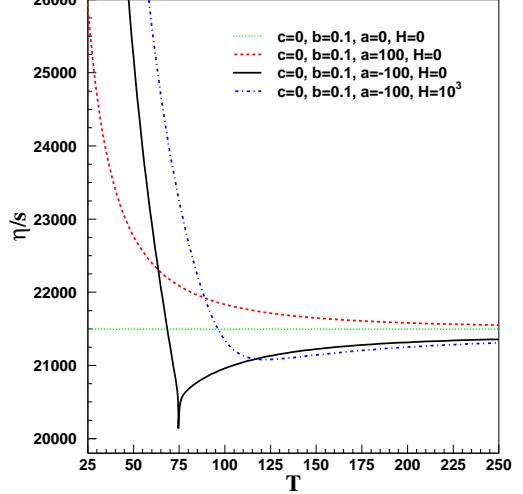


FIG. 1: η/s vs. T for cases with a second-order phase transition (solid curve), a crossover (dash-dotted curve), and with no phase transition for massive field (dashed curve) and massless field (dotted curve). Parameters can be in arbitrary units.

The entropy density of the system is given as $s = -\partial V(\phi_0)/\partial T$, while the shear viscosity η is calculated using the Boltzmann equation. It is proven that in a weakly coupled scalar field theory with quartic and cubic terms, summing the leading order diagrams for η is equivalent to solving the Boltzmann equation with effective T dependent mass and scattering amplitudes [25]. Thus, one can directly apply the Boltzmann equation to compute η for cases with $c = 0$ for both the symmetric ($a > 0$) and symmetry breaking ($a < 0$) cases. Furthermore, since the proof of Ref. [25] does not use properties that are restricted to scalar theories, the conclusion is expected to hold for more general theories with weak couplings, including QCD in the perturbative regime [26]. Here, we also apply it to the $c > 0$ case with a first-order phase transition.

The two-particle elastic scattering amplitude, which governs particle collisions in the Boltzmann equation, is

$$i\mathcal{T} = \lambda_4 + \lambda_3^2 \left[\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right], \quad (7)$$

where s, t and u are Mandelstam variables, and $\lambda_3 = 6\phi_0(b + \frac{10c}{3}\phi_0^2 + 10cL(\phi_0))$ and $\lambda_4 = 12(\frac{b}{2} + 5c\phi_0^2 + 5cL(\phi_0))$ are effective couplings.

Now we discuss the behavior of η/s , starting from the simplest case: $a = c = 0$ and $b > 0$. In this theory, the only scale in the problem is T and the system is always in the symmetric phase. Thus, both η and s are proportional to T^3 on dimensional ground and η/s is T independent (up to the logarithmic running of b which we neglected).

When the mass term is added, η/s is no longer a con-

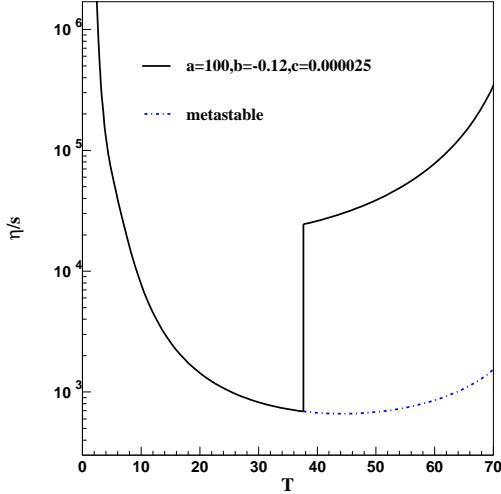


FIG. 2: η/s vs. T for the ground state (solid curve) and metastable state (dash-dotted curve) in a first-order phase transition. Parameters can be in arbitrary units.

stant. η/s curves for $b > 0$, $c = 0$ and (i) $a > 0$ (ii) $a < 0$ are shown in Fig. 1. (We have used $b = 0.1$. The expansion parameter is $\sim \lambda/(4\pi)^2$, where $\lambda = 6b$. [27]) For (i), there is no phase transition. η/s is always monotonically decreasing. For (ii), η/s reaches its minimum and develops a cusp at the T_c of the second-order phase transition. There is clearly a qualitative difference in the η/s behavior between cases with and without a phase transition. In fact, in the high T expansion ($bT^2 \gg |a|$),

$$\frac{\eta}{s} = \frac{k}{b^2} \left(1 + \frac{0.24}{\sqrt{b}} \frac{a}{T^2} + \dots \right), \quad (8)$$

with $k = 192 + 49\sqrt{b} + \mathcal{O}(b)$, consistent with η computed in [25, 28]. Thus, the sign of a determines whether η/s is decreasing or increasing at high T . As $T \rightarrow 0$, s approaches zero exponentially (the excitations are massive) while η approaches zero by power laws. Thus, η/s is decreasing in both cases at low T . (This feature is not affected by the Hartree approximation used at low T .) One concludes that η/s is not monotonic in T for a second-order phase transition. Although our mean-field result would be modified very close to T_c (with $|T - T_c|/T_c \lesssim \mathcal{O}(b)$), we argue that the most natural scenario for η/s in a second-order phase transition is to have a single local minimum at T_c (where m also reaches its minimum, $m = 0$), because there is no other T that is more special than the others to develop another minimum in η/s . Analogously, the most natural scenario for η/s with no phase transition is monotonic decreasing because no T is more special than the others to develop a minimum. Finally, for a second-order phase transition, one naturally expects a cusp at T_c , while the cusp is smoothed out in a crossover ($H \neq 0$). These “natural-

ness” arguments do not depend on whether the coupling is weak. In fact, there is no b that is more special than the others such that one does not expect the η/s behavior to change in the strong coupling cases ($|b| \gg 1$) either.

For the first-order phase transition ($a > 0$, $b < 0$, and $c > 0$), the result for η/s is shown in Fig. 2. There is a discontinuity at T_c , as expected. The η/s is concave in general and its minimum is reached at T_c from the low T side. The qualitative behaviors above and below T_c are similar to those of a second-order phase transition. The main difference is the discontinuity. It is not intuitively obvious why this should be the case because even though the effective potentials of the first- and second-order phase transitions look similar near the minima (thus the masses behave similarly) the Lagrangians are still quite different. It turns out this is simply because the scattering amplitude of Eq.(7) is dominated by the λ_4 term in both cases as long as $T^2 > |a|$. Also, the fact that the minimum of η/s is reached at T_c from below, instead of above, is physical but nontrivial. It is physical, because particle interaction is stronger in the low temperature side. It is nontrivial, because the discontinuities in η and s are of the same sign. So it is not obvious the discontinuity of η/s has to be governed by η .

The above-mentioned behaviors of η/s in first-, second-order phase transitions and crossover are the same as those in H_2O , N , He , and all the matters with data available in the NIST database [9, 21, 22]. We expect the agreement still holds when the scalar field is generalized to N components and the Lagrangian has an $O(N)$ symmetry. This is because the arguments used to explain the η/s behaviors do not depend on N , except for Eq.(8) in the $c = 0$ case. However, in the $O(N)$ theory, the prefactors in Eq.(8) change [29] but the signs remain. Thus, the arguments still hold.

Given the general agreement in the η/s behaviors of H_2O , He , N , $O(N)$ scalar field theories, and QCD and cold atoms in some known cases, it is conceivable that the η/s behaviors shown in Figs. 1 and 2 are general properties of a large class of fluids, including QCD and cold atoms. If this is correct, then the best place to measure the minimum η/s to test the minimum bound conjecture is near T_c . In particular, for a pure gluon theory which has a first-order phase transition, the minimum η/s could be just below T_c . This strongly motivates future lattice computations to be carried out in the lower T regime. In addition, this also provides theoretical support for locating QCD CEP by measuring η/s . There is some subtlety regarding the CEP. In H_2O , He , or N , a cusp in η/s is observed at the CEP with experimental resolution. Theoretically, it is argued that η has a mild divergence (and hence η/s) at QCD CEP [30] as in the Model H according to Hohenberg and Halperin’s classification [31] (but see [32] for a caveat). Experimentally, divergence of η in 3He liquid-gas transition critical point was observed [33] but not for the tricritical point of 3He - 4He binary fluid [34]. Both cases belong to Model H and were predicted to have the same weak divergence by Ref. [31]. In any

case, it is unlikely that RHIC has the sensitivity to this effect.

Finally, we comment on the counterexamples of the minimum η/s bound conjecture constructed in [16, 17]. A common feature of those counterexamples is a large number of flavor g . In the weak coupling and small density limits, one can compute η (also using the Boltzmann equation) and s reliably. In the non-relativistic examples employed in [16, 17], particle numbers are fixed or controlled by chemical potentials. It was found that η does not scale with g (because it is normalized to the density already) but s does. Thus, $\eta/s \propto 1/(\text{Log}(g)a_s^2)$ as $g \rightarrow \infty$ (a_s is the two-body scattering length) and the $1/4\pi$ bound was violated. There is an important difference in those counterexamples on whether flavor changing is allowed during the collisions. In the $O(N)$ scalar field theory, $\phi_1\phi_1 \rightarrow \phi_i\phi_i$ ($i = 1, 2, \dots, N$) collision can happen. Thus, it is a model with flavor changing. To the leading order in the b expansion, $\eta/s \propto 1/(Nb^2)$ for $a = c = 0$, similar to Eq.(8) but with an extra factor of N from s . Naively the conjecture is violated when $N \rightarrow \infty$. However, the expansion parameter is (Nb) . One needs $b \propto 1/N$ or smaller (and hence $\eta/s \propto N$ or larger) to be consistent with the weak coupling treatment. Thus, one cannot conclude that the $O(N)$ scalar field theory violates the bound in the large N limit based on weak coupling argument. The same is true to the non-relativistic flavor changing counterexamples of [16, 17] ($a_s \propto 1/g$ is required). In the strong coupling regime, the $b \propto 1/N$ scaling seems not necessary. However, without the scaling, the β function of b blows up in the large N limit. It is not clear whether a sensible non-perturbative theory can be constructed this way. Our example sets no constraint on flavor conserving cases in [16, 17], though.

In Ref. [16], there is discussion about whether the η/s bound conjecture should be applied to metastable

states. In the definition using Kubo's formula, η is related to the linear response of an ensemble. It is hard to exclude the ground state from the ensemble to compute the metastable state η . A similar problem happens in computing η with the Boltzmann equation in which the dissipation of perturbations away from thermal equilibrium gives rise to η . The ground state information is encoded in the equilibrium distribution. However, experimentally it makes perfect sense to measure η of a material as long as it is stable during the time of the measurement. Water is a sharp example of this [16]. In electroweak interaction, water is metastable to proton fusion of the two hydrogen atoms despite the large Coulomb barrier. However, one can modify the Lagrangian by sending the weak boson masses to infinity to stabilize water. The resulting η will be close to the experimentally measured value. It is not clear whether the same result can be obtained by simply discarding near ground states from the ensemble. Assuming this is correct, we investigate its consequence in our model by expanding ϕ around the false vacuum above the T_c of a first-order phase transition. As shown in Fig. 2, η/s of the metastable state can be lower than the minimum η/s of the true ground state (the effect will be more pronounced if c is bigger). The metastable state η/s has the tendency to keep decreasing until this state eventually disappears at higher T . It would be interesting to see whether this behavior persists in the strongly interacting theory.

We thank Tom Cohen, Yusuke Nishida, Dirk Rischke, Dam Son and Larry Yaffe for useful discussions. MH thanks the National Taiwan U. for hospitality. JWC thanks the U. of Washington for hospitality. This work is supported by the IHEP, Chinese Academy of Sciences, CAS key project KJCX3-SYW-N2, NSFC10735040, and the NSC and NCTS of Taiwan.

[1] K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333.
 [2] M. A. Stephanov, PoS **LAT2006**, 024 (2006) [arXiv:hep-lat/0701002].
 [3] P. Jacobs *et al.*, arXiv:0705.1930 [nucl-ex].
 [4] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, Nature **443**, 675 (2006).
 [5] M. Asakawa and K. Yazaki, Nucl. Phys. A **504**, 668 (1989); M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998).
 [6] R. A. Lacey *et al.*, Phys. Rev. Lett. **98**, 092301 (2007); arXiv:0708.3512.
 [7] P. Kovtun, D.T. Son, and A.O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
 [8] T. Schafer, arXiv:cond-mat/0701251; G. Rupak and T. Schafer, arXiv:0707.1520 [cond-mat.other].
 [9] L. P. Csernai, J. I. Kapusta and L. D. McLerran, Phys. Rev. Lett. **97**, 152303 (2006).
 [10] J. W. Chen and E. Nakano, Phys. Lett. B **647**, 371 (2007).
 [11] I. Arsene *et al.*, Nucl. Phys. A **757**, 1 (2005); B. B. Back *et al.*, *ibid.* **757**, 28 (2005); J. Adams *et al.*, *ibid.* **757**, 102 (2005); K. Adcox *et al.*, *ibid.* **757**, 184 (2005).
 [12] D. Molnar and M. Gyulassy, Nucl. Phys. A **697**, 495 (2002) [Erratum *ibid.* **703**, 893 (2002)].
 [13] D. Teaney, Phys. Rev. C **68**, 034913 (2003).
 [14] P. Romatschke and U. Romatschke, arXiv:0706.1522.
 [15] A. Nakamura and S. Sakai, Phys. Rev. Lett. **94**, 072305 (2005). H. B. Meyer, arXiv:0704.1801 [hep-lat].
 [16] T. D. Cohen, Phys. Rev. Lett. **99**, 021602 (2007) [arXiv:hep-th/0702136]. A. Cherman, T. D. Cohen and P. M. Hohler, arXiv:0708.4201 [hep-th].
 [17] A. Dobado and F. J. Llanes-Estrada, Eur. Phys. J. C **51**, 913 (2007) [arXiv:hep-th/0703132].
 [18] Y. Kats and P. Petrov, arXiv:0712.0743 [hep-th].
 [19] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, Phys. Rev. D **77**, 126006 (2008) [arXiv:0712.0805 [hep-th]]; Phys. Rev. Lett. **100**, 191601 (2008) [arXiv:0802.3318 [hep-th]].
 [20] X. H. Ge, Y. Matsuo, F. W. Shu, S. J. Sin and T. Tsukioka, arXiv:0808.2354 [hep-th].

- [21] J. W. Chen, Y. H. Li, Y. F. Liu and E. Nakano, arXiv:hep-ph/0703230.
- [22] E.W. Lemmon *et al.*, Thermophysical Properties of Fluid Systems, in *NIST Chemistry WebBook*, Eds. Linstrom P.G. & Mallard, W.G., March 2003 (<http://webbook.nist.gov>).
- [23] J.M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).
- [24] J. T. Lenaghan and D. H. Rischke, J. Phys. G **26**, 431 (2000) [arXiv:nucl-th/9901049].
- [25] S. Jeon, Phys. Rev. D **52**, 3591 (1995); S. Jeon and L. Yaffe, Phys. Rev. D **53**, 5799 (1996).
- [26] P. Arnold, G. D. Moore and L. G. Yaffe, JHEP **0305**, 051 (2003); JHEP **0011**, 001 (2000).
- [27] A. G. Cohen, D. B. Kaplan and A. E. Nelson, Phys. Lett. B **412**, 301 (1997) [arXiv:hep-ph/9706275].
- [28] G. D. Moore, arXiv:0706.3692 [hep-ph].
- [29] G. Aarts and J. M. Martinez Resco, JHEP **0402**, 061 (2004) [arXiv:hep-ph/0402192].
- [30] D. T. Son and M. A. Stephanov, Phys. Rev. D **70**, 056001 (2004) [arXiv:hep-ph/0401052].
- [31] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49** (1977) 435.
- [32] K. Ohnishi, K. Fukushima and K. Ohta, Nucl. Phys. A **748**, 260 (2005) [arXiv:nucl-th/0409046].
- [33] C.C. Agosta, S. Wang, L.H. Cohen, H. Meyer, J. Low Temp. Phys. **67**, 237 (1987).
- [34] C. Howald, X. Qin, H. S. Nham, and H. Meyer, J. Low Temp. Phys. **86**, 375 (1992).